Bellerophon: Tactical Theorem Proving for Hybrid Systems

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Cyber-Physical Systems

Cyber-Physical Systems combine computation and control.

Hybrid Systems model combinations of discrete and continuous dynamics.

Bellerophon

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Verifying hybrid systems is hard.
Bellerophon demonstrates how to tackle hybrid systems with tactics:

- Build on a sound core.
- Implement high-level primitives for hybrid systems proofs.
- Automate common constructions (for ODEs and control software)

Bellerophon

Theorem	Bellerophon LOC	Conceptual Proof Steps	Hybrid Systems Axiom Applications
Static Safety	12	71	30,355
Passive-Friendly Safety	45	140	68,620
Orientation Safety	15	108	173,989
Pass Intersection Liveness	234	440	61,878

KeYmaera X: Trustworthy Foundations

Interactive Reachability Analysis

- ➢ Bellerophon combinator language
- ➢ Bellerophon standard library for hybrid systems
- ➢ Demonstration

Bellerophon for Automation and Tooling

Conclusions & Resources

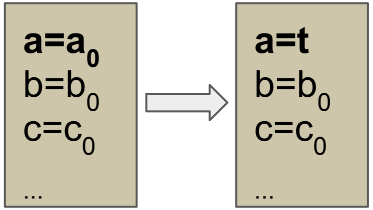
Trustworthy Foundations

KeYmaera X enables trustworthy automation for hybrid systems analysis:

- A well-defined **logical foundations,**
- implemented in a **small trustworthy core**
- that ensures correctness of **automation** and **tooling**.

Hybrid Programs

$a := t$



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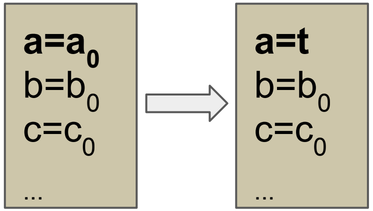
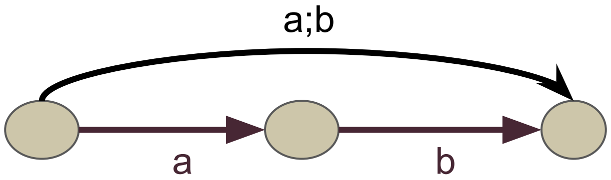


a;b



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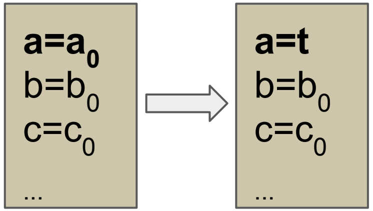

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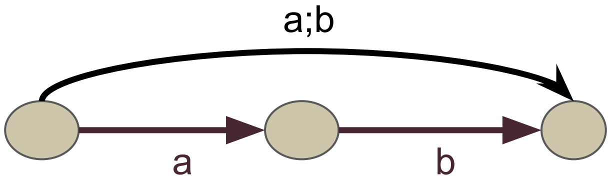

?P

If P is true: no change

If P is false: terminate

Hybrid Programs

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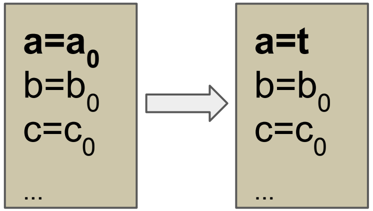

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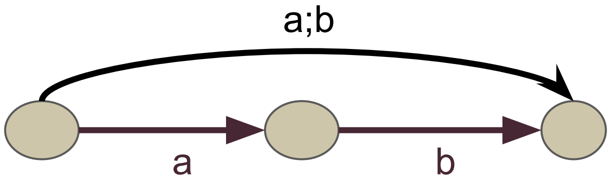

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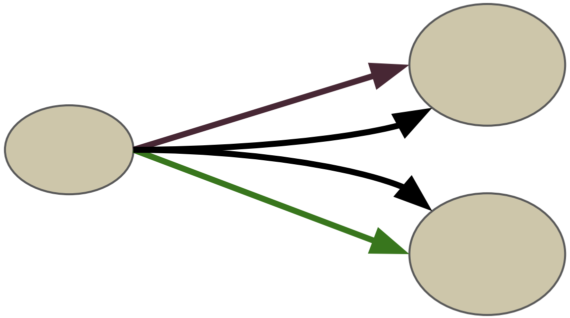

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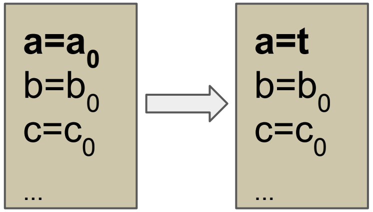

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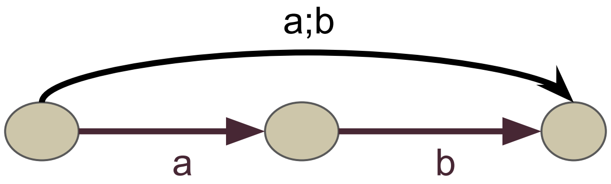

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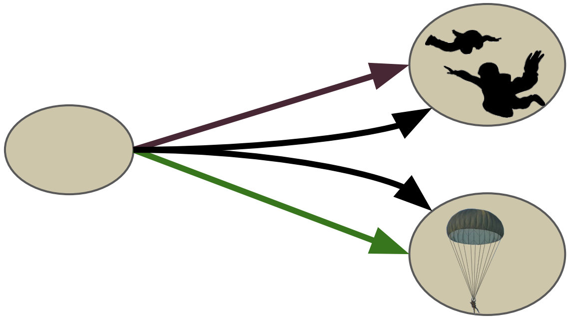

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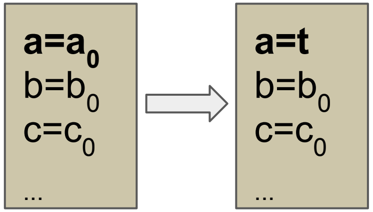

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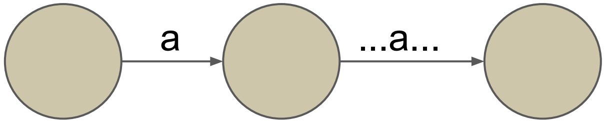

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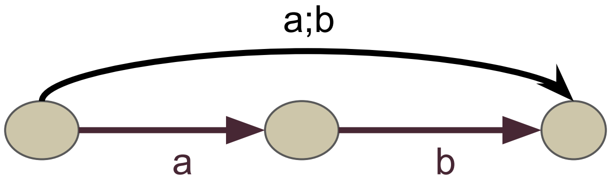
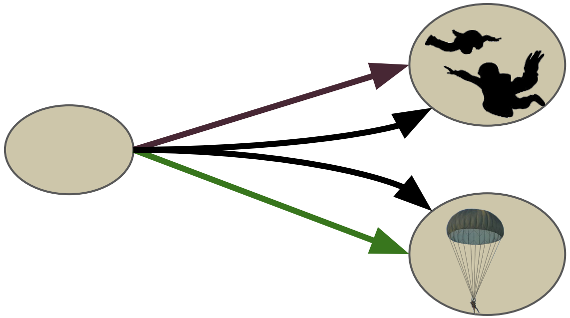


a ∪ b



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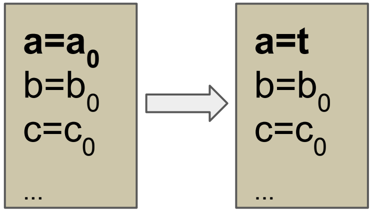

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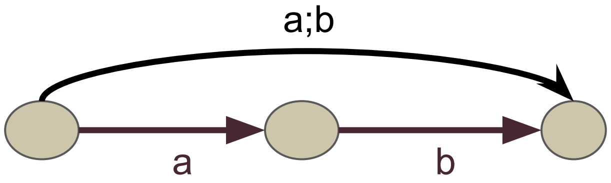

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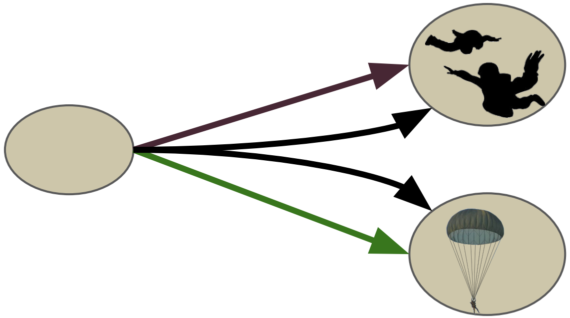

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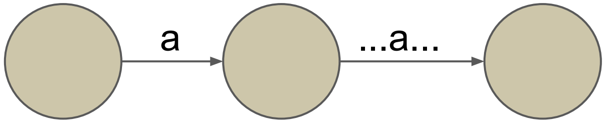
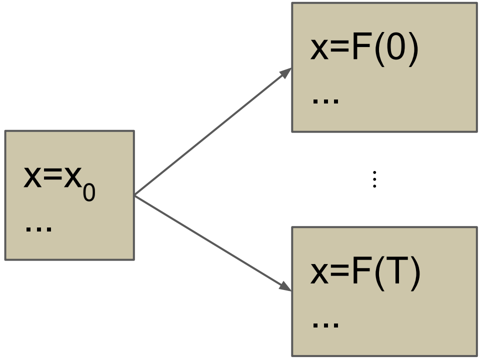


x'=f



Reachability Specifications

[a]P	“after every execution of a, P”
<a>P	“after some execution of a, P”

Reachability Specifications

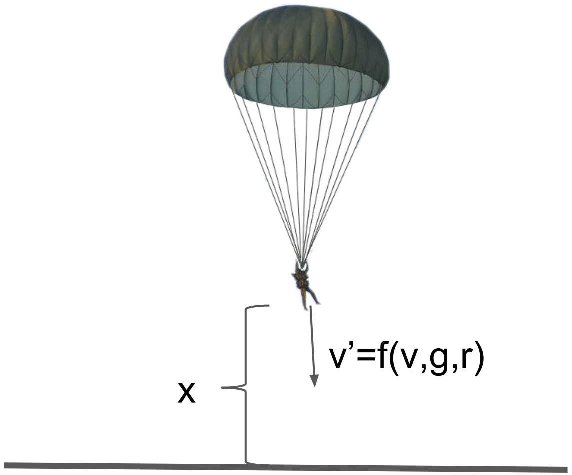
$[a]P$ “after every execution of a, P”

$\langle a \rangle P$ “after some execution of a, P”

$$init \rightarrow [\{x := u(x);\ x' = f(x)\}^*]safe$$

Hello, World

```
{
  {?Dive ∪ r := r_p};
  t:=0;
  {x' = v,
   V' = f(v,g,r), t'=1
   & 0≤x & t≤T}
}*
```



Control: Continue diving if safe, else open parachute.

Plant: Downward velocity determined by gravity, air resistance.

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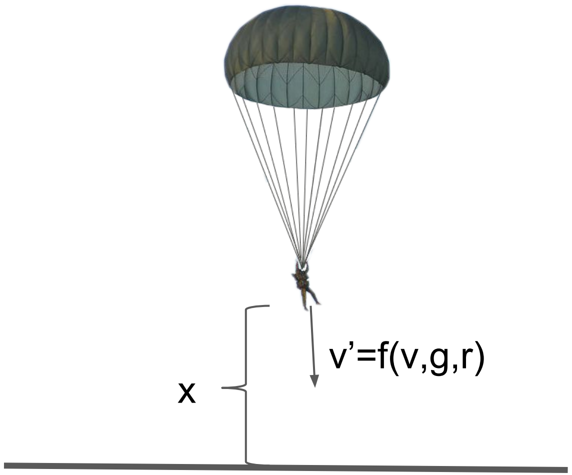

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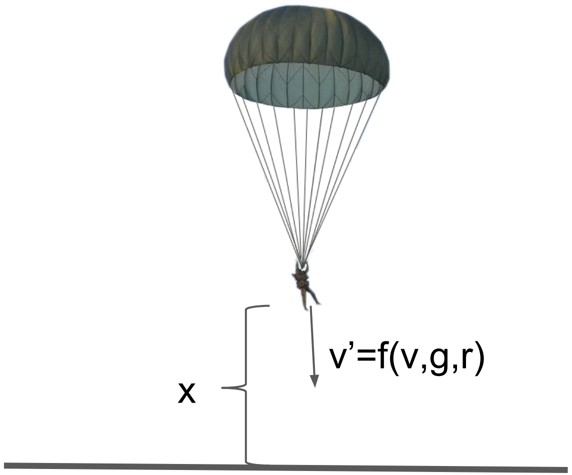

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{
   {?Dive ∪ r := r_p};
   t:=0;
   {x′ = v,
    V′ = f(v,g,r), t′=1
    & 0≤x & t≤T}
}*
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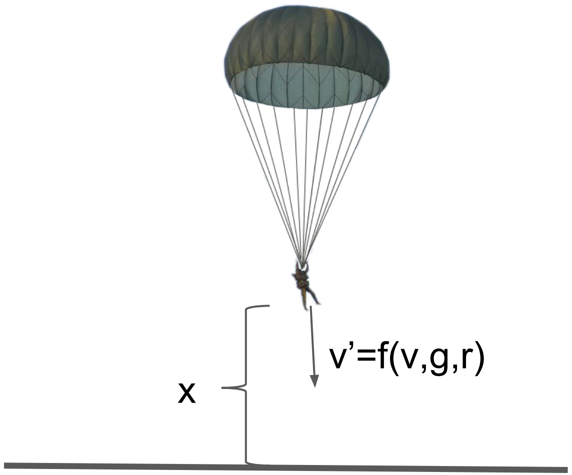

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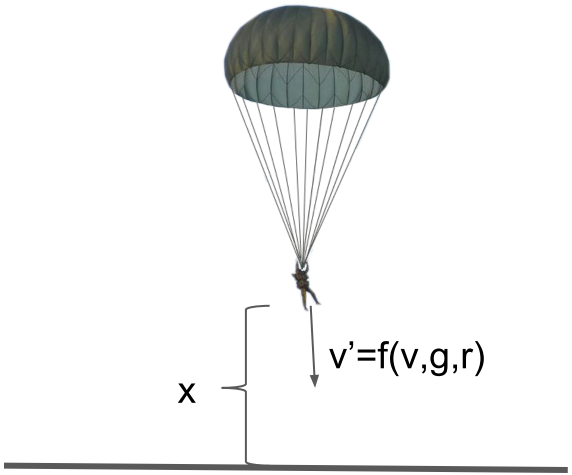

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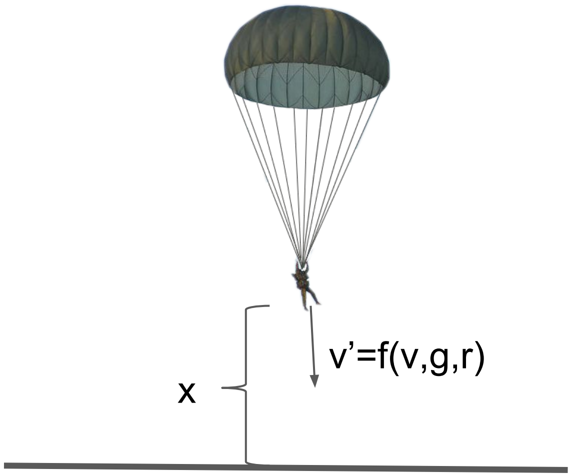

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(Dive & g>0 & ...)→
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}*] (x=0→m≤v)
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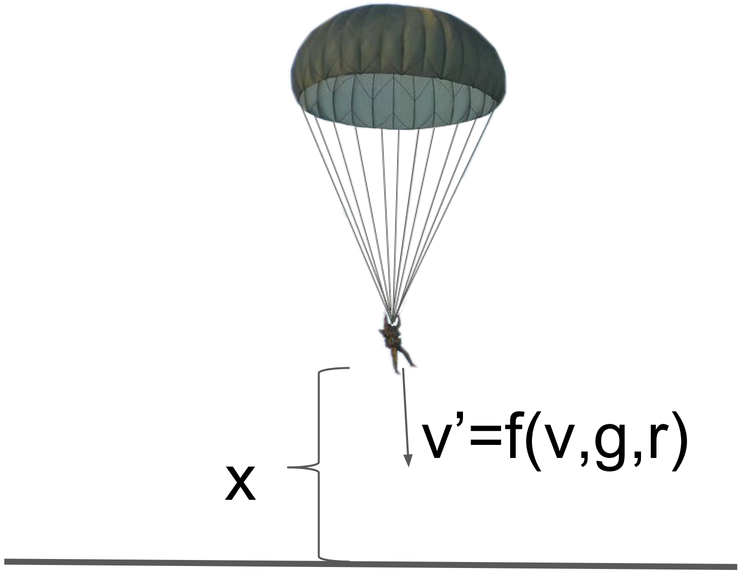

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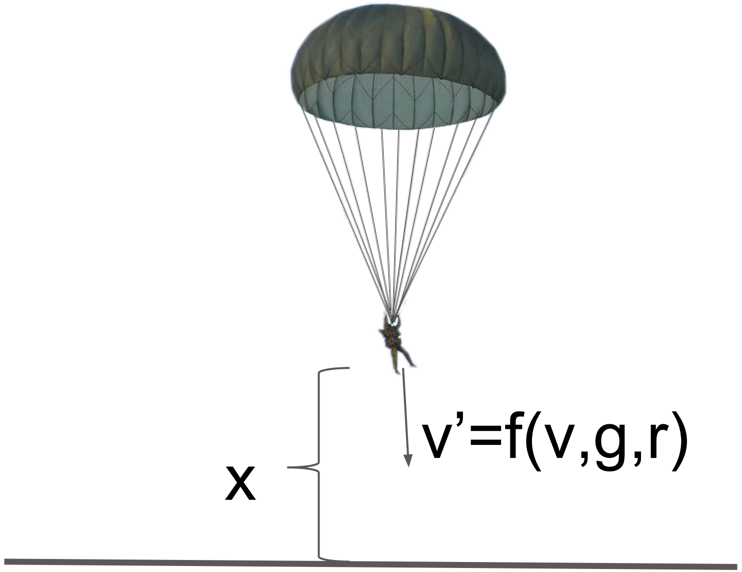
If the parachuter is on the ground, their speed is safe ($m \le v \le 0$)

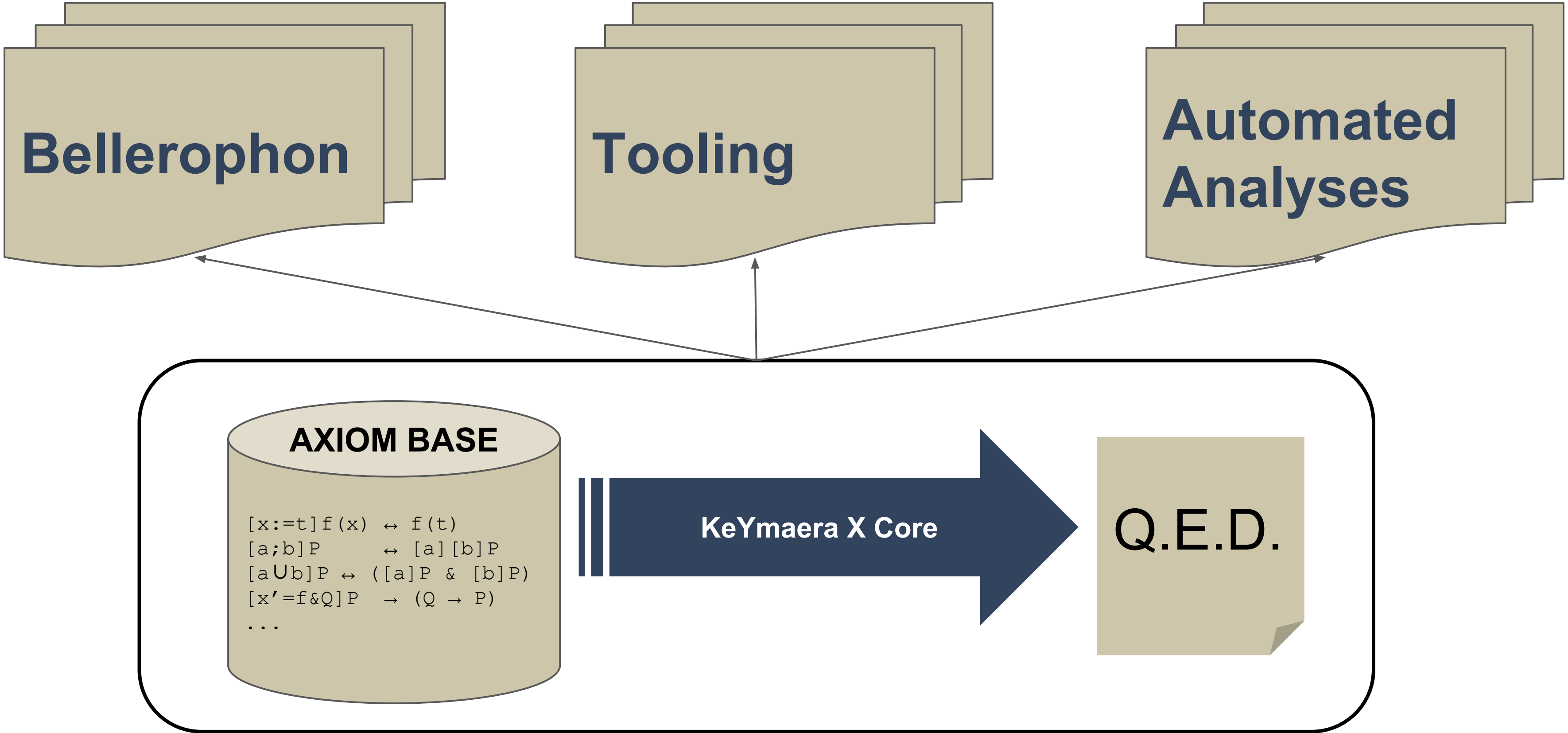


Dynamical Axioms

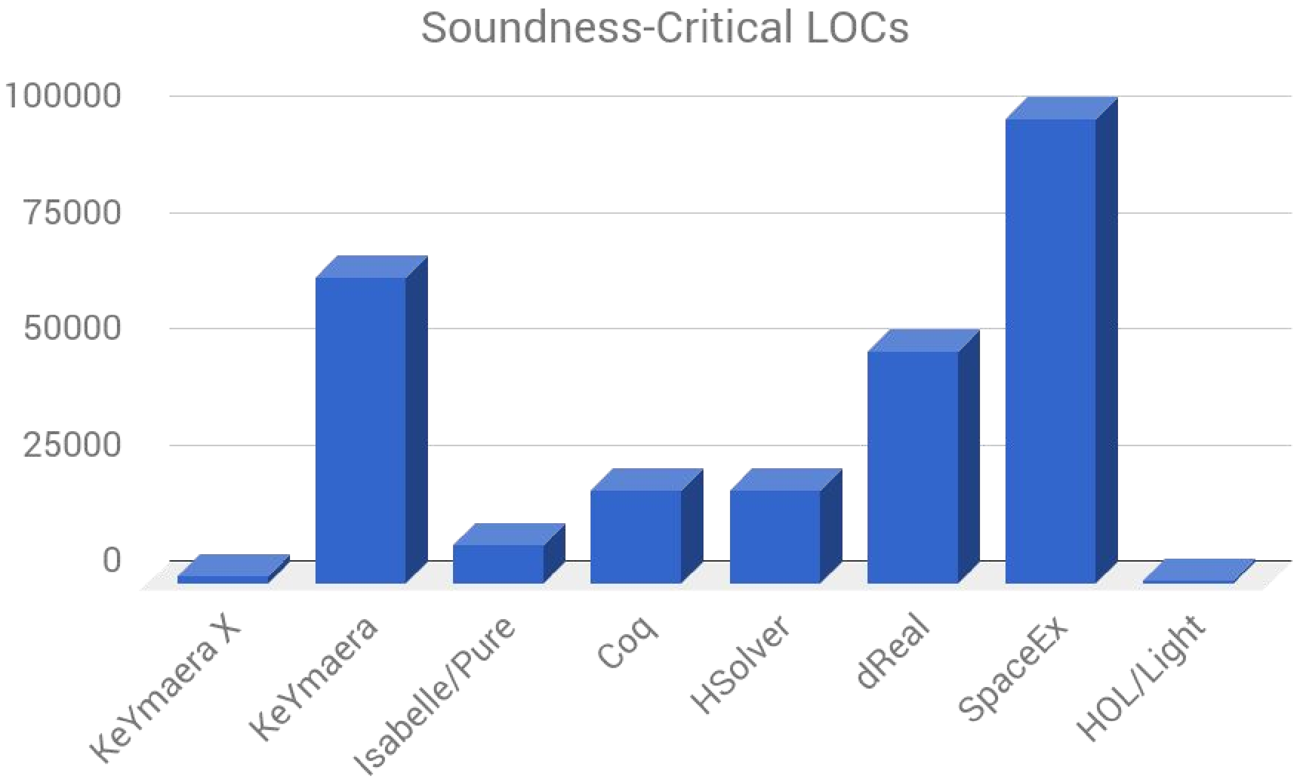
```
[x:=t]f(x)  ↔  f(t)
[a;b]P      ↔  [a][b]P
[a∪b]P      ↔  ([a]P & [b]P)
[x'=f&Q]P   →  (Q → P)
...
```

Trusted Core

Trustworthy Implementations



Prover Core Comparison



Bellerophon

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Bellerophon enables interactive verification and tool development:

- A **standard library** of common proof techniques.
- A **combinator language/library** for **decomposing** theorems and **composing** proof strategies.

Bellerophon

Standard Library

Tactic	Meaning
`prop`	Applies propositional reasoning exhaustively.
`unfold`	Symbolically executes discrete, loop-free programs.
`loop(J, i)`	Applies loop invariance axiom to position i.
`dI,dG,dC,dW`	Reasoning principles for differential equations.

Bellerophon

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1000+

Bellerophon

Combinators

1000+

Tactic	Meaning
`prop`	Applies propositional reasoning exhaustively.
`unfold`	Symbolically executes discrete, loop-free programs.
`loop(J, i)`	Applies loop invariance axiom to position i, extends J with constants.
`dI,dG,dC,dW`	Reasoning principles for differential equations.

Combinator	Meaning
A ; B	Execute A on current goal, then execute B on the result.
A | B	Try executing A on current goal. If A fails, execute B on current goal.
A*	Run A until it no longer applies.
$A<(B_1,B_2, \ldots ,B_N)$	Execute A on current goal to create N subgoals. Run B_i on subgoal i.

Isolating Interesting Questions

(Dive & g>0 & …)→

[{

}*] **(x=0→m≤v)**

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prop ; loop(J,1)

(Dive & g>0 & …) → J

Loop invariant holds initially

J → [

] J

Loop invariant is preserved

J → x=0→m≤v

Loop invariant implies safety

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Bellerophon

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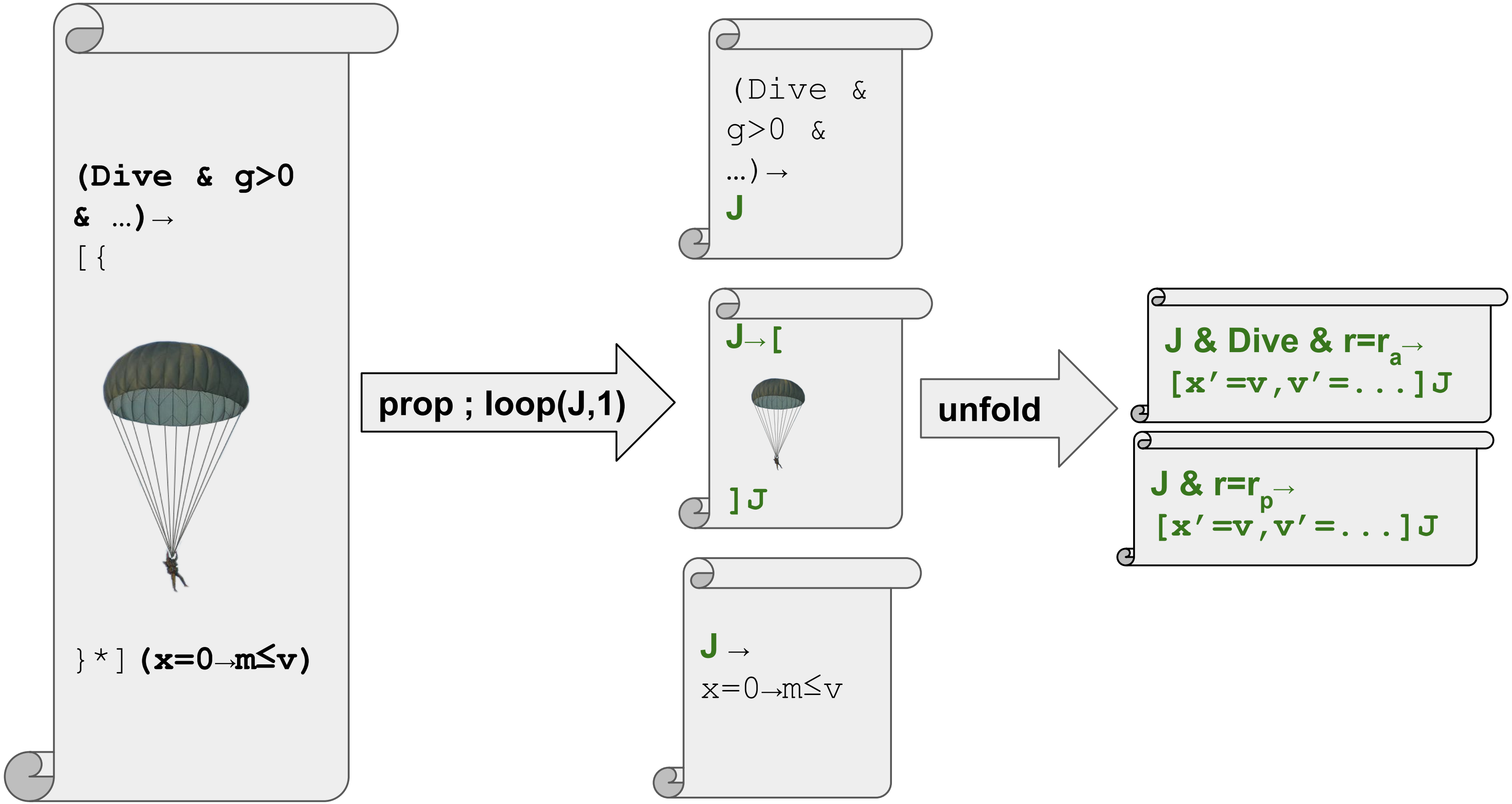

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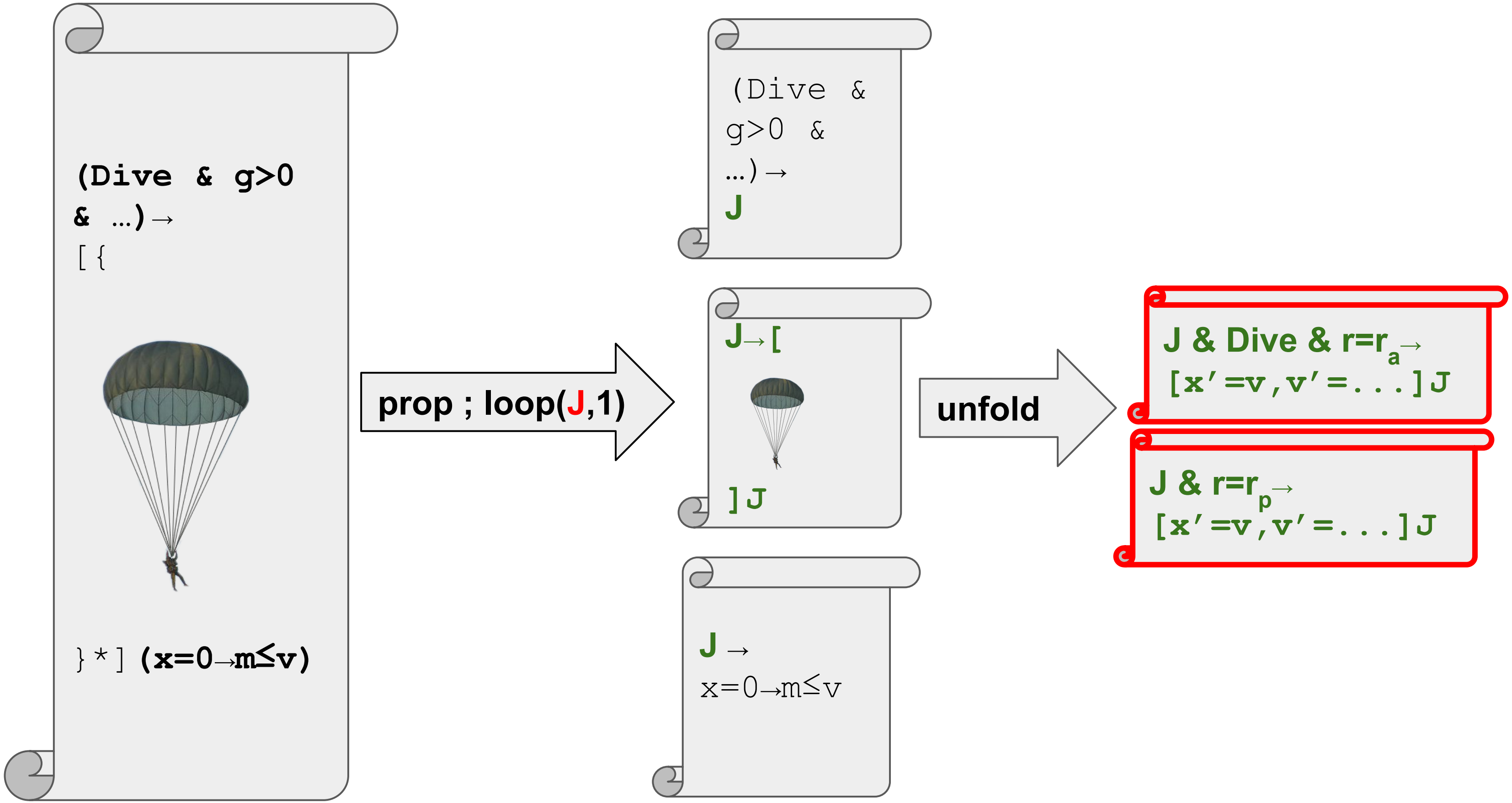


Isolating Interesting Questions

```
prop ;  loop(J, 1) <(
   QE, /* Real arith. solver */
   QE,
   unfold ; <(
      … /* parachute open case */
      … /* parachute closed case */
   )
)
```

Trustworthy Standard Library at High Abstraction Level

```
J → [{ctrl; plant}*]J
```

J = v > -sqrt(g/pr) > m & …

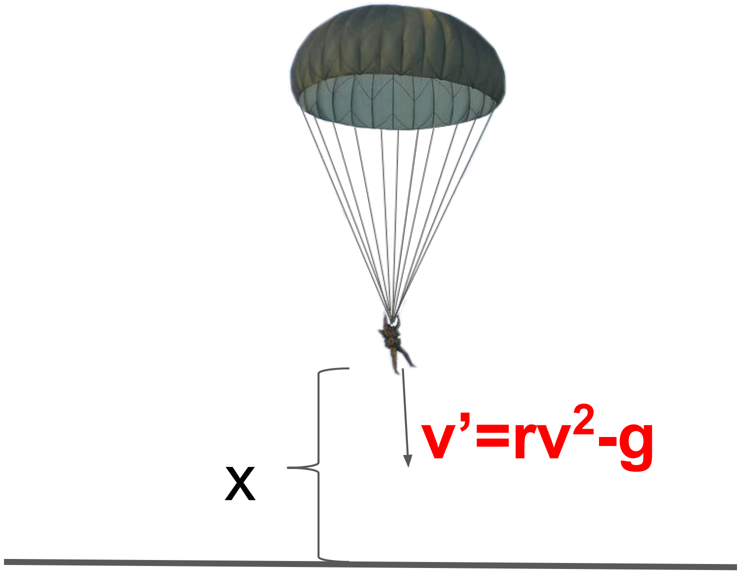
Parachute Open Case:

$v \geq v_0 - gt$

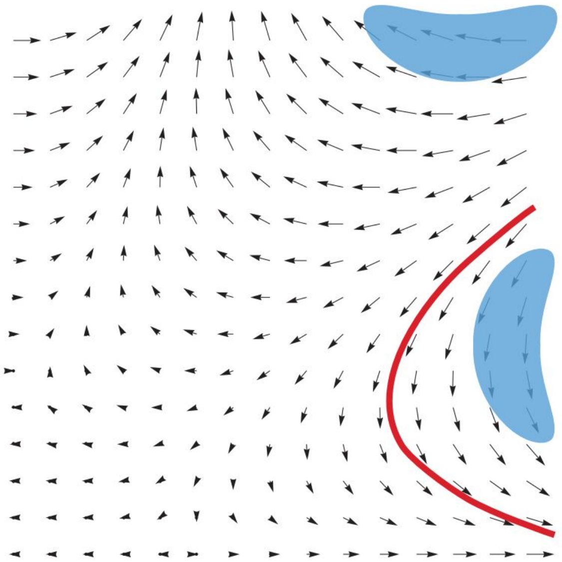
$\geq v_0 - gT$

$> -\mathrm{sqrt}(g/pr)$

Inductive invariants



From Axioms to Proof Steps



DI Axiom:

[{x'=f&Q}]P↔([?Q]P←(Q→[{x'=f&Q}]P'))

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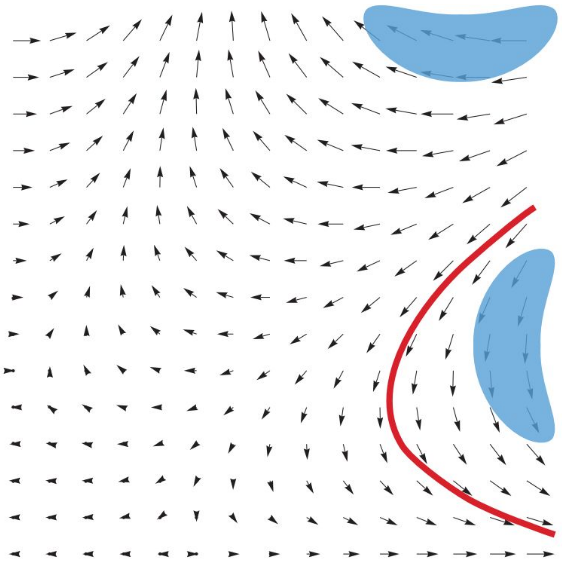

DI Axiom:

$[\{x'=f\&Q\}]P \leftrightarrow ([?Q]P \leftarrow (Q \rightarrow [\{x'=f\&Q\}]P'))$

Example:

$[v'=r_p v^2-g, t'=1] v \geq v_0 - gt$

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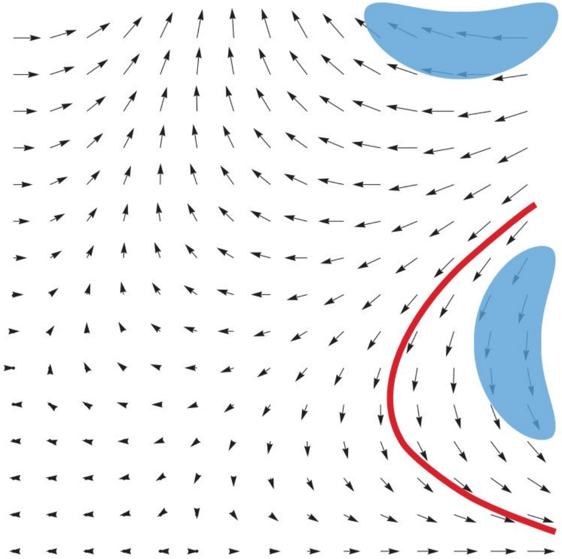

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Example:

$[v'=r_p v^2-g, t'=1]v \geq v_0 - gt$ ↔

… ↔

$[v':=r_p v^2-g][t':=1]v' \geq -g*t'$ ↔

$r_p v^2-g \geq -g$ ↔

$r_p \geq 0$

From Axioms to Proof Steps

dI Tactic:

Side derivation:

$(v \geq v_0 - gt)' \leftrightarrow$

$(v)' \geq (v_0 - gt)' \leftrightarrow$

$(v)' \geq (v_0 - gt)' \leftrightarrow$

$(v)' \geq (v_0)' - (gt)' \leftrightarrow$

$(v)' \geq (v_0)' - (t(g)' + g(t')) \leftrightarrow$

$v' \geq v_0' - (tg' + gt')$

$H = r_p \geq 0 \ \& \ r_a \geq 0 \ \& \ g > 0 \ \& \ \ldots$

DI Axiom:

$[\{x'=f \& Q\}]P \leftrightarrow ([?Q]P \leftarrow (Q \rightarrow [\{x'=f \& Q\}]P'))$

Example:

$[v'=r_p v^2-g, t'=1] v \geq v_0 - gt \leftrightarrow$

… $\leftrightarrow$

$[v' := r_p v^2-g][t' := 1] v' \geq -g * t' \leftrightarrow$

$r_p v^2-g \geq -g \leftrightarrow$

$H \rightarrow r_p \geq 0$

Automation and Tooling

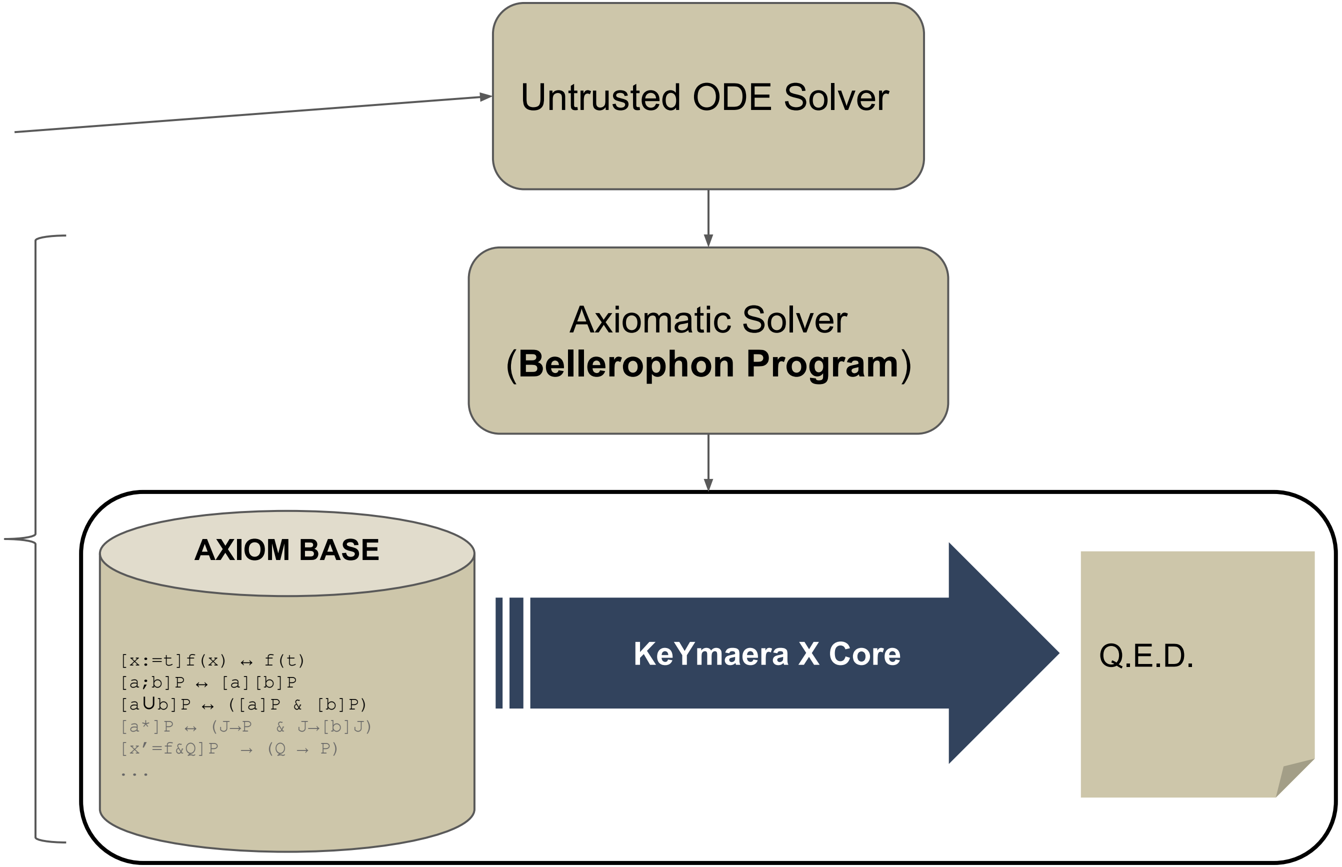
Hybrid Systems Analyses can be built on top of KeYmaera X.

Examples:

- ODE Solver
- Runtime Monitoring

Solving Differential Equations

1. Use untrusted code to find a conjecture.
2. Prove the conjecture systematically, leveraging standard library.



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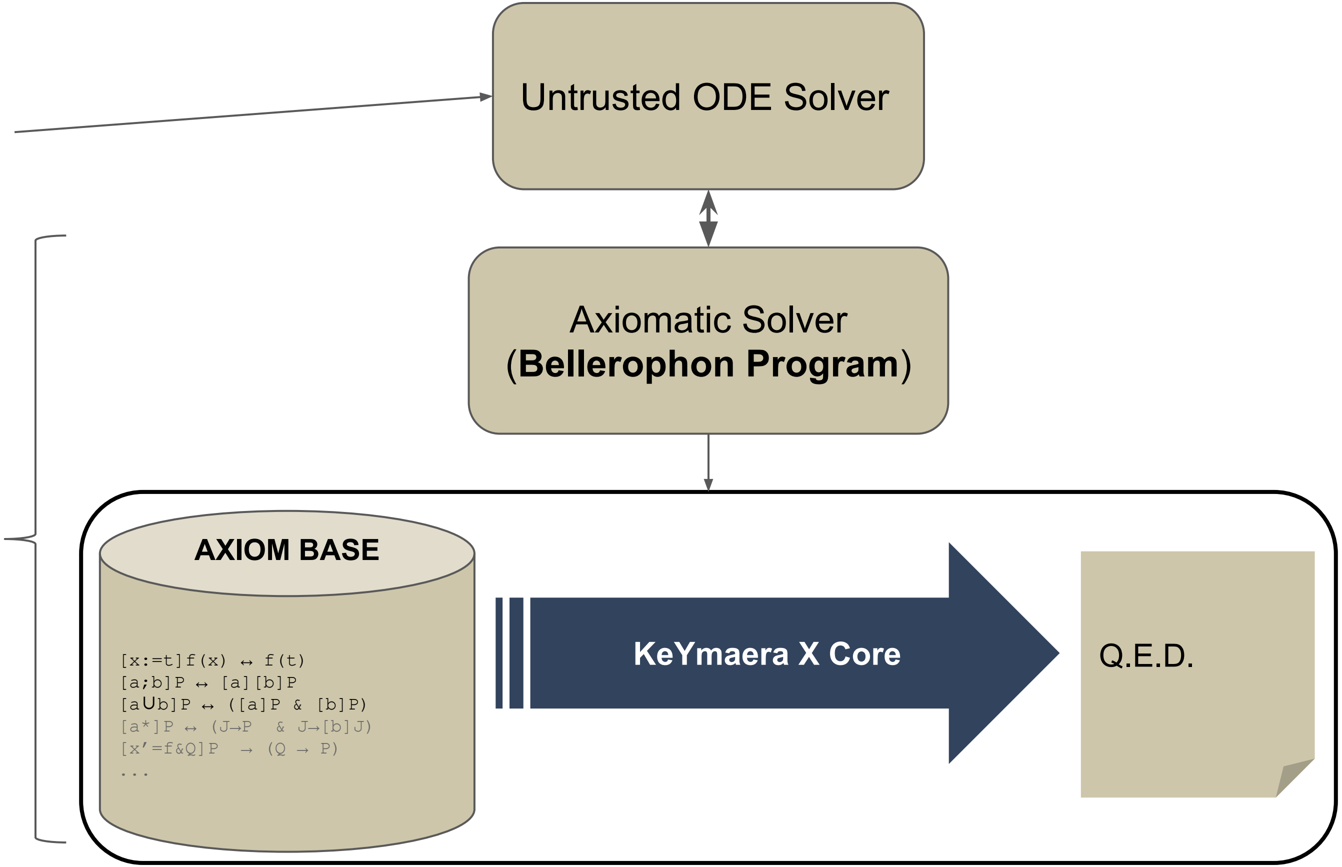
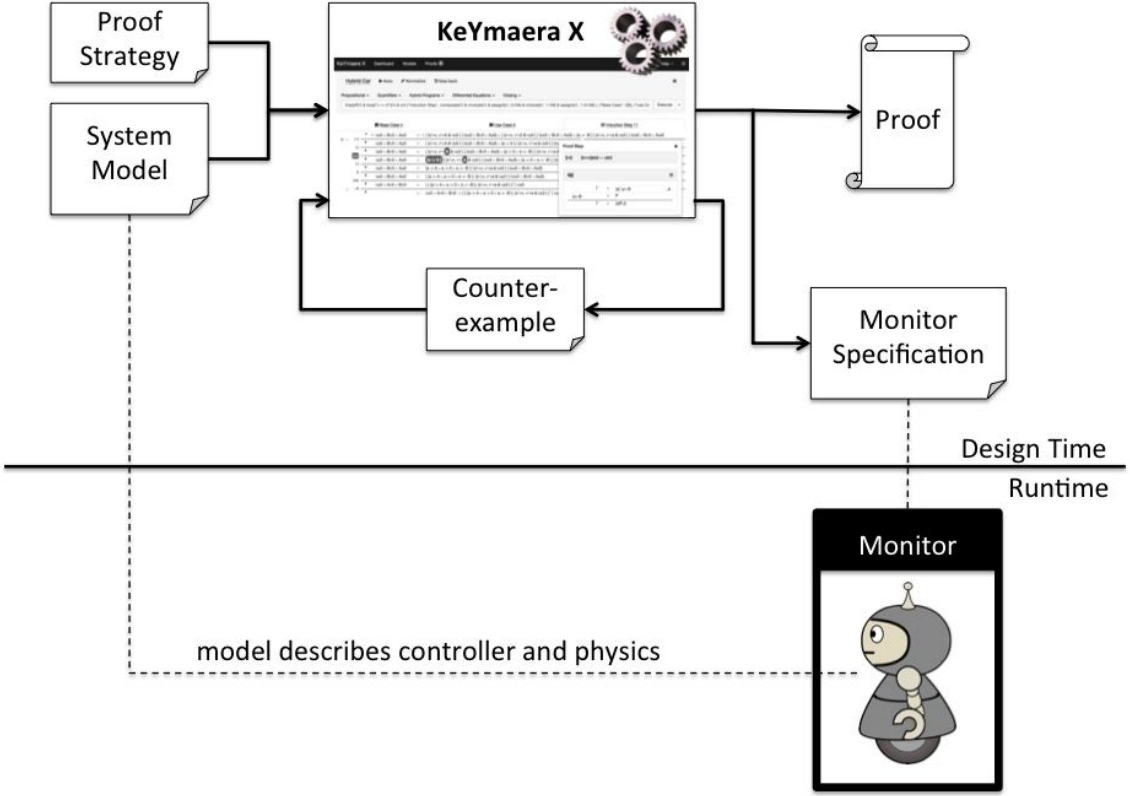


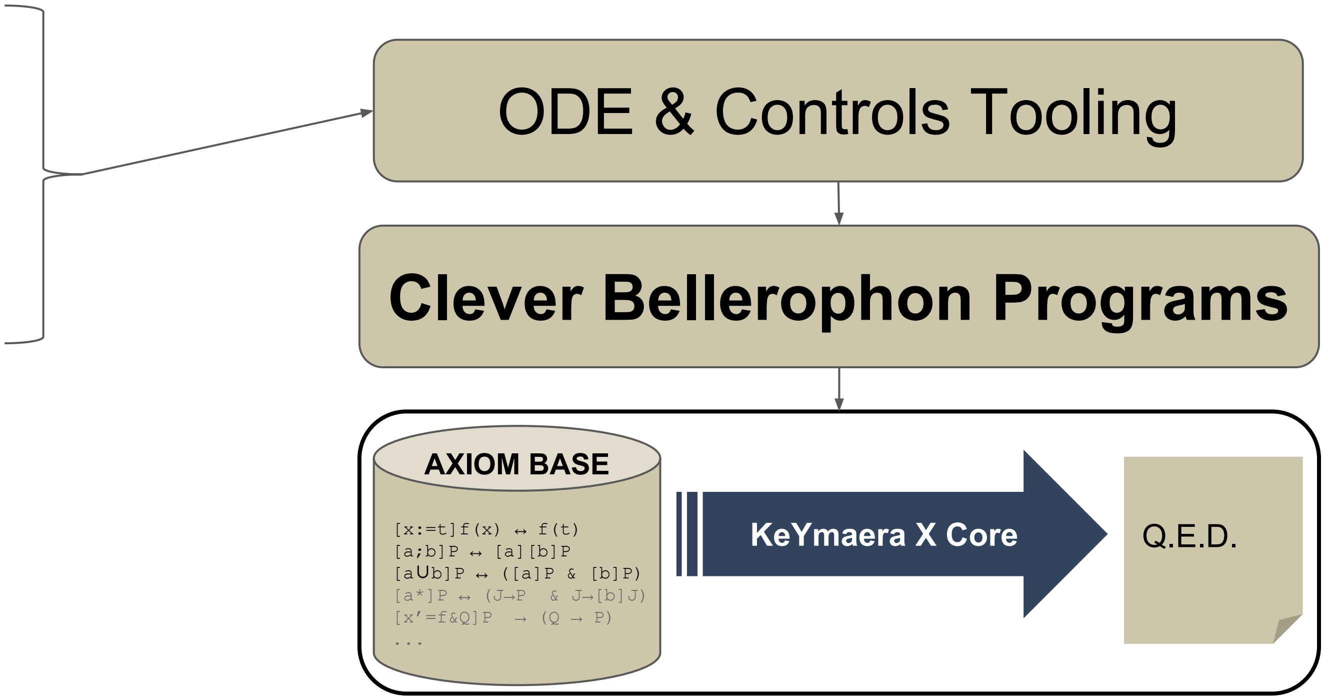
Automation and Tooling

ModelPlex Tactic



Other Proof Automation & Tooling

- **Taylor Series**
- **Bifurcations**
- Limit Cycles
- Numerical tools
- ...



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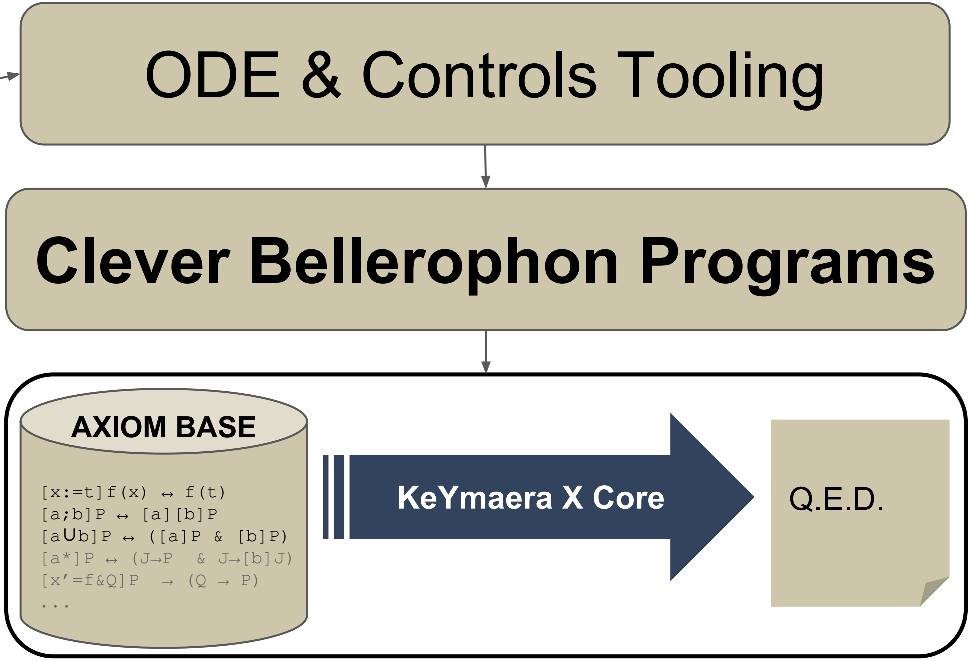

Other Tooling:

- Component-based Verification
- Web UI

Conclusion

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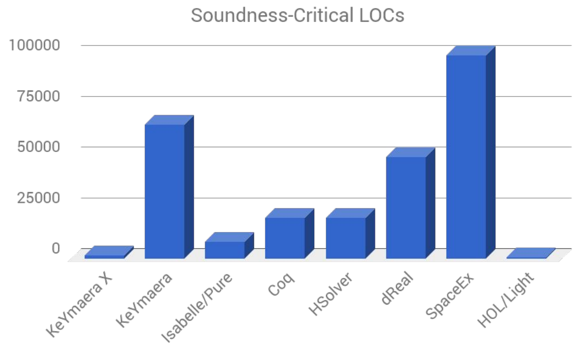

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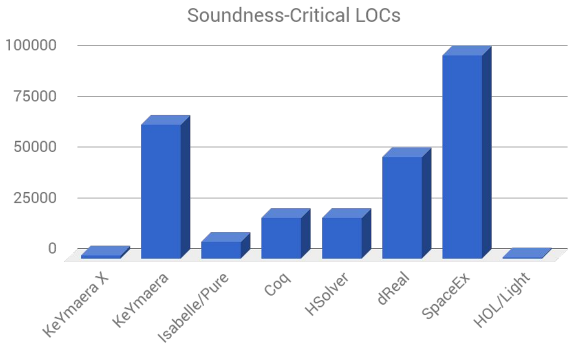
Belleroph on demonstrates how to verify hybrid systems using tactics.



dI Tactic:

Side derivation:
$(v \geq v_0 - gt)' \leftrightarrow$
$\ldots \leftrightarrow$
$\ldots \leftrightarrow$
$\ldots$

$H = r_p \geq 0$ & $r_a \geq 0$ & $g > 0$ & ...

DI Axiom:
$[\{x'=f\&Q\}]P \leftrightarrow ([?Q]P \leftarrow (Q \rightarrow [\{x'=f\&Q\}]P'))$

Example:
$[v'=r_p v^2-g, t'=1]v \geq v_0 - gt \quad \leftrightarrow$
$\ldots \quad \leftrightarrow$
$[v':=r_p v^2-g][t':=1]v' \geq -g*t' \quad \leftrightarrow$
$r_p v^2-g \geq -g \quad \leftrightarrow$
$H \rightarrow r_p \geq 0$

Conclusion

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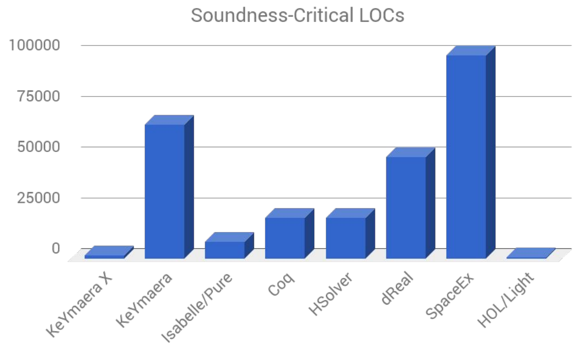

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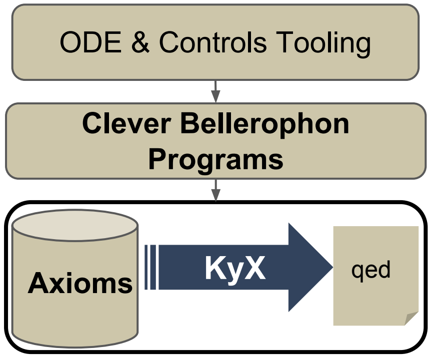

Conclusion

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Bellerophon demonstrates how to verify hybrid systems using tactics.

Project Website (start here)	keymaeraX.org
Online Demo	web.keymaeraX.org
Open Source (GPL)	github.com/ls-lab/KeYmaeraX-release

Thanks: 15-424 students, **Jean-Baptiste Jeannin**, Khalil Ghorbal, **Yanni Kouskoulas** et al., and many others!

Developers:

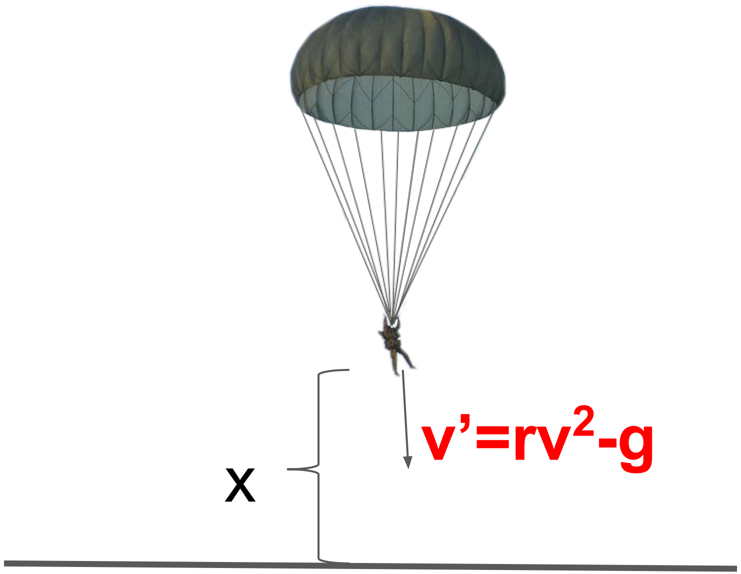
- Stefan Mitsch
- Nathan Fulton
- André Platzer
- Brandon Bohrer
- Jan-David Quesel
- Yong Kiam Tan
- Markus Völp

Differential Ghosts

Parachute Closed:

$$J \ \& \ t=0 \ \& \ r=r_p \ \rightarrow$$
$$[x'=v, v'=rv^2-g \ \& \ 0 \leq x \ \& \ t \leq T] v>-\mathrm{sqrt}(g/pr) \ > \ m$$

Proof requires a **differential ghost** because the property is **not inductive**.



Differential Ghosts

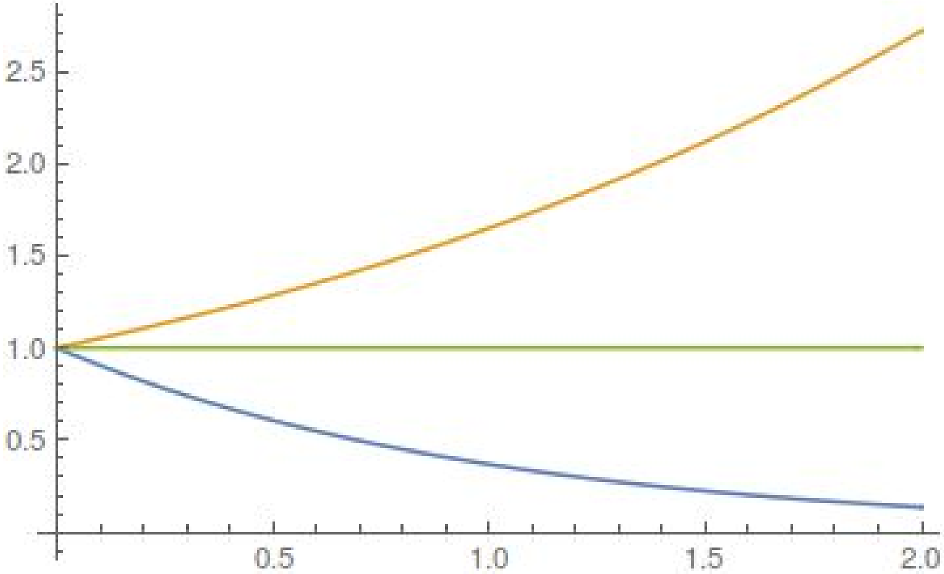
An example differential ghost.

$$x>0 \rightarrow [x'=-x]x>0$$

Differential Ghosts

An example differential ghost.

```
x>0 → [x'=-x]x>0
Ghost:      y'=y/2
Conserved: 1=xy²
```



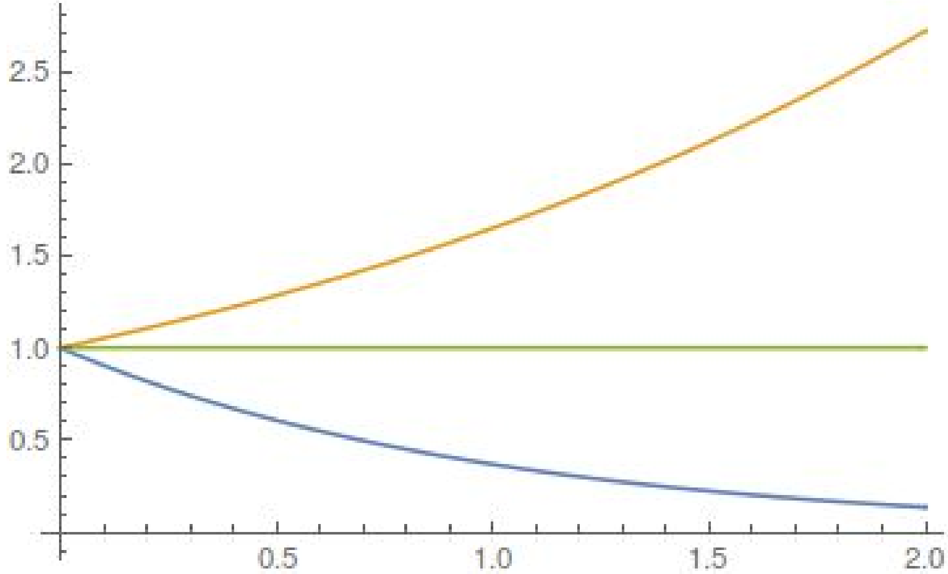
Differential Ghosts

An example differential ghost.

$x>0 \rightarrow [x'=-x]x>0$

Ghost: $y'=y/2$

Conserved: $1=xy^2$



Notice:

$x>0 \leftrightarrow \exists y. 1=xy^2$

Therefore, suffices to show:

$1=xy^2 \rightarrow \exists y. [x'=-x, y'=y/2] 1=xy^2$

Prover Core Comparison

Tool	Trusted LOC (approx.)
KeYmaera X	**1,682 (out of 100,000+)**
KeYmaera	65,989
Isabelle/Pure	8,113
Coq	20,000
HSolver	20,000
dReal	50,000
SpaceEx	100,000